

# A Framework for Systematic Study of QCD Vacuum Structure II: Coherent Lattice QCD

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Jul 20 2006

## Abstract

We propose the formulation of lattice QCD wherein all elements of the theory (gauge action, fermionic action,  $\theta$ -term, and all operators) are constructed from a single object, namely the lattice Dirac operator  $D$  with exact chiral symmetry. Several regularizations of this type are suggested via constructing scalar densities (gauge actions) that are explicit functions of  $D$ . The simplest of these is based on the proposition that classical limit of density associated with  $\text{Tr } D$  is (up to an additive constant) proportional to  $F^2$ , while the corresponding operator is local. The possibilities of explicit interrelations between gauge and fermionic aspects of the theory are emphasized together with the utility of such formulations for exploring the QCD vacuum structure.

## 1 Introduction

This is the second in the series of papers whose purpose is to introduce the set of ideas and approaches aimed at creating a systematic framework for studying QCD vacuum structure in the path integral formalism [1]. The basic premise of such framework is the use of lattice QCD both to define the relevant representation of the vacuum (the associated ensemble of Euclidean QCD), as well as an exclusive source of necessary dynamical information obtained via numerical simulation (the *Bottom-Up* approach). One of the important points emphasized in Ref. [1] is that in the search for the *fundamental structure* (the structure relevant for all aspects of QCD physics), it is beneficial to utilize the freedoms we have in the lattice definition of QCD. In particular, one can take advantage of the fact that the degree of space-time order in typical configurations, as measured by Kolmogorov entropy, can vary significantly over the set of valid lattice theories at a given cutoff. In order to reach the theories with high degree of space-time order, we have proposed to use the transformations in the set of gauge configurations, *chiral orderings* [1], whose meaning is to replace a given link  $U_{n,\mu}$  with the effective SU(3) phase  $\bar{U}_{n,\mu}$  associated with hopping of chiral fermion from site  $n + \mu$  to site  $n$ . More precisely, it was proposed that the configuration-based deformation of the gauge action performed via chiral ordering will lead to the evolution in the set of actions that largely preserves the physical content of the lattice theory (*principle of chiral ordering*) while lowering its Kolmogorov entropy.

The strategy of searching in the set of actions via chiral ordering transformations takes advantage of the fact that, for purposes of the space–time structure, it is not necessary to know the form of the underlying action explicitly. Indeed, all we need is the access to its ensemble (“typical configurations”). However, it would certainly be beneficial if it would be possible to work with actions that generate a high degree of space–time order and, at the same time, can be explicitly written down. In this paper we suggest the class of actions that might satisfy this requirement. To do so, we will adopt a characteristic feature of the chiral ordering approach, namely that the transformed gauge actions become functions of the lattice Dirac kernel  $D$  on which the transformation is based. In other words, we propose gauge actions that are explicit functions of  $D$ . While this does not in itself guarantee that the Kolmogorov entropy in these theories will be significantly lower than those of standard actions (e.g. Wilson gauge action), we think this is a very reasonable expectation. Moreover, the theories so obtained will facilitate various interrelations between gauge and fermionic aspects of the theory and are interesting in their own right. We emphasize (as we did in Ref. [1]) that our approach acquires its full potential only in the framework of QCD with dynamical fermions. Definition of full theory based on a single object, namely a lattice Dirac kernel  $D$  describing chiral fermion, is perhaps a cleanest manifestation of this fact. In what follows, we will refer to any lattice regularization based on this idea as *coherent* lattice QCD. Also, “lattice QCD” will frequently be abbreviated as LQCD.

Upon accepting the possibility of coherent LQCD, one realizes rather quickly that there is a large freedom of choice available here. We explore this freedom to some extent by searching for a formulation that casts gauge and fermionic contributions into an overall dynamics of the theory in the most form–symmetric manner. In the resulting regularization, *symmetric logarithmic* LQCD, the gluonic contribution can be viewed exactly as that of an infinitely heavy quark. Conversely, the full effective action can be expressed as a sum of  $N_f + 1$  gluon–like terms with only one being local (representing the usual gluon), and the rest of them non–local. The meaning of non–local  $F^2$  is that for smooth configurations the operator effectively averages  $F^2$  over the physical distance associated with renormalized quark mass in question. The feasibility of symmetric logarithmic LQCD rests on the resolution of certain locality issues and the ways of approaching them are discussed in detail.

One can alternatively arrive at coherent LQCD via formal/classical considerations in the continuum. Indeed, it can be argued that the full action for a classical configuration (smooth almost everywhere) can be written entirely in terms of Dirac operator  $D$ . The particular way in which the corresponding formal equation comes about suggests a prescription for lattice regularization of QCD once lattice Dirac operator is specified. We refer to this class of lattice theories as *classically coherent* LQCD due to the important role of classical fields in obtaining it.

In order to make the lattice theory fully coherent, it is desirable that the operators used for measuring physical observables are also explicit functions of  $D$ . This, in fact, has already been achieved in a generic way via the use of a chiral ordering transformation [1]. However, it would be useful if at least certain important local composite fields could be expressed in yet more explicit way. We discuss in some detail the possibilities offered by Clifford decomposition of  $D$ .

One of the interesting aspects of coherent LQCD is that it admits a natural definition of *effective* LQCD at fermionic response scale  $\Lambda_F$  along the lines discussed in [1]. Let us recall

that the purpose of effective LQCD is to study the *effective structure* of QCD vacuum, where the influence of fluctuations above a certain scale is suppressed. Effective structure defines an “unfolding” of all-encompassing fundamental structure into the one that is relevant for processes that “excite” mainly fluctuations down to some desired length scale. Effective LQCD at fixed fermionic response scale  $\Lambda_F$  is the first step in such a definition. The feature of coherent LQCD is that, contrary to the generic case, it allows for definition of effective LQCD in an explicit manner, i.e. via the action that can be written down and, at least in principle, directly simulated. This is discussed in the last part of the paper.

## 2 Coherent Lattice QCD

In this section we discuss the simplest version of coherent LQCD, where the gauge action is constructed in direct analogy with representation of topological term using lattice Dirac operator with exact chiral symmetry. As a basis for such construction we propose the following conjecture.<sup>1</sup>

**Conjecture C3.** *Let  $A_\mu(x)$  be arbitrary smooth  $\text{su}(3)$  gauge potentials on  $\mathbb{R}^4$ . If  $U(a) \equiv \{U_{n,\mu}(a)\}$  is the transcription of this field to the hypercubic lattice with classical lattice spacing  $a$ , and  $\mathbb{I} \equiv \{U_{n,\mu} \rightarrow \mathbb{I}^c\}$  is the free field configuration then*

$$\text{tr} \left( D_{0,0}(U(a)) - D_{0,0}(\mathbb{I}) \right) = -c^S a^4 \text{tr} F_{\mu\nu}(0) F_{\mu\nu}(0) + \mathcal{O}(a^6) \quad (1)$$

for generic  $D \in \mathcal{S}^F$ . Here  $c^S$  is a non-zero constant independent of  $A_\mu(x)$  at fixed  $D$ , and  $F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]$  is the field-strength tensor.

We wish to make the following remarks for completeness.<sup>2</sup>

(i) By potentials being “smooth” we mean differentiable arbitrarily many times. Relation (1) is expected to be valid also for configurations with singular gauge potentials  $A_\mu(x)$  if they are smooth in some finite neighborhood of  $x = 0$ .

(ii) Transcription to the hypercubic lattice with classical lattice spacing  $a$  is defined via

$$U_{n,\mu}(a) \equiv \mathcal{P} \exp \left( a \int_0^1 ds A_\mu(an + (1-s)a\hat{\mu}) \right) \quad (2)$$

where  $\mathcal{P}$  is the path ordering symbol and  $\hat{\mu}$  a unit vector in direction  $\mu$ .

(iii) The set  $\mathcal{S}^F$ , defined in Ref. [1], contains single-flavor lattice Dirac operators  $D$  (or equivalently actions  $\bar{\psi} D \psi$ ) satisfying the usual requirements including the exact lattice chiral symmetry.

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<sup>1</sup>We will keep the notation consistent with the first paper of the series (Ref. [1]) to the largest extent possible. The numbering of conjectures and definitions is also continued. Conventions for continuum gauge theory are summarized in Appendix A for convenience.

<sup>2</sup>It is desirable to specify and fix our convention for denoting traces. The basic rule is that  $\text{tr}$  denotes a local trace, i.e. trace of a matrix at given  $x$  or  $n$ , while  $\text{Tr}$  denotes a global trace, i.e. trace involving space-time coordinates. In the above form, the traces are implicitly assumed to be taken in the linear space “natural” to the object in question, e.g. in Eq. (1) the trace on left-hand side is over spin-color, while the trace on the right-hand side is over color only. If it is still necessary to distinguish spin and color traces, we use the corresponding superscripts, e.g.  $\text{tr}^s$ ,  $\text{tr}^c$ .

(iv) Due to the translation invariance of  $D$  it is sufficient to consider the statement of Conjecture C3 only at  $x = n = 0$ . Analogous statement is true for arbitrary fixed  $x$ .

(v) The heuristic reason for validity of Conjecture C3 is that  $\text{tr} D_{n,n}$  is scalar, local, gauge invariant operator (function of the gauge field). Up to dimension four, the possibilities for such operators in the continuum only include a constant and  $F_{\mu\nu}F_{\mu\nu}$ .

(vi) The relation analogous to (1) for the pseudoscalar case [2] (see also Ref. [3]), namely

$$\text{tr} \gamma_5 \left( D_{0,0}(U(a)) - D_{0,0}(\mathbb{I}) \right) = -c^P a^4 \text{tr} F_{\mu\nu}(0) \tilde{F}_{\mu\nu}(0) + \mathcal{O}(a^6) \quad (3)$$

is a basis for definition of topological charge density via chirally symmetric Dirac operator. Here  $\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$  (with  $\epsilon_{1234} = 1$ ) is the dual of the field-strength tensor, and the constant term on the left-hand side vanishes.

(vii) For the family of standard overlap Dirac operators [4], based on Wilson-Dirac operator with mass  $-\rho$ , the existence of constant  $c^P(\rho)$  has been verified in explicit calculations [5] and is given by <sup>3</sup>

$$\frac{c^P(\rho)}{\rho} = \frac{1}{8\pi^2} \quad 0 < \rho < 2 \quad (4)$$

The validity of Conjecture C3 will be supported numerically in Ref. [6], where the constants  $c^S(\rho)$  for the family of overlap Dirac operators will be determined.

(viii) In practice, we frequently consider QCD in finite physical volume and the statement analogous to Conjecture C3 can be verified most directly in such setting. The relevant conclusion can be formulated as follows.

**Conjecture C3a.** *Let  $A_\mu(x)$  be arbitrary smooth  $\text{su}(3)$  gauge potentials on symmetric torus of size  $L_p$ , and let  $a_L \equiv L_p/L$  be the classical lattice spacing on  $L^4$  hypercubic lattice. If  $U(a_L) \in \mathcal{U}^L$  is the transcription of  $A_\mu(x)$  to this lattice then the following non-zero finite limit exists*

$$\lim_{L \rightarrow \infty} \frac{\text{tr} \left( D_{0,0}(U(a_L)) - D_{0,0}(\mathbb{I}) \right)}{a_L^4 \text{tr} F_{\mu\nu}(0) F_{\mu\nu}(0)} = -c^S \quad (5)$$

for generic  $D \in \mathcal{S}^F$ . Here  $c^S$  is a constant independent of  $A_\mu(x)$  at fixed  $D$ .

Due to the locality of  $D$ , the constants  $c^S$  in Conjectures C3 and C3a are expected to be identical.

We can now define the action for simplest version of coherent LQCD with  $N_f$  flavors of quarks and the CP-violating  $\theta$ -term as

$$S_{\bar{\beta}, \bar{\theta}, \{m_f\}} = \text{Tr} (\bar{\beta} - i\bar{\theta}\gamma_5) (D(U) - D(\mathbb{I})) + \sum_{f=1}^{N_f} \bar{\psi}^f (D(U) + m_f) \psi^f \quad (6)$$

where  $\{m_f\}$  is the set of real non-negative quark masses, and <sup>4</sup>

$$\bar{\beta}(\beta) = \frac{\beta}{12c^S} \quad \left( \beta \equiv \frac{6}{g^2} \right) \quad \bar{\theta}(\theta) = \frac{\theta}{16\pi^2 c^P} \quad \left( \theta \in (-\pi, \pi) \right) \quad (7)$$

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<sup>3</sup>Note that the “standard” overlap operator is given by  $D^{(\rho)} = \rho[1 + A(A^\dagger A)^{-\frac{1}{2}}]$  where  $A = D_W - \rho$  and  $D_W$  is the massless Wilson-Dirac operator.

<sup>4</sup>We should perhaps emphasize here that everything in this equation, including the bare parameters, is in lattice units. See also the notational conventions set in Ref. [1].

Indeed, the locality of  $S$  defined above follows from locality of  $D$ , the required symmetries follow from its transformation properties, and the classical correspondence is equivalent to Eqs. (1) and (3). Note that the free-field term in the gauge part of the action only contributes a field-independent constant and can be discarded.

The dynamics of coherent QCD (6) is completely encoded in the chirally symmetric lattice Dirac operator. This feature appears particularly clearly after fermionic variables are integrated out, which leads to the distribution density of the gauge fields given by

$$P_{\bar{\beta}, \bar{\theta}, m} \propto e^{\text{Tr}[N_f \ln(D+m) + (-\bar{\beta} + i\bar{\theta}\gamma_5)D]} = \det \left[ (D+m)^{N_f} e^{(-\bar{\beta} + i\bar{\theta}\gamma_5)D} \right] \quad (8)$$

where  $D \equiv D(U)$ , and we consider lattice QCD with  $N_f$  mass-degenerate flavors of mass  $m$ . This can also be written as

$$\begin{aligned} P_{\bar{\beta}, \bar{\theta}, m} &\propto \det \left[ e^{i\bar{\theta}\gamma_5(D+m)} \left( -\frac{d}{d\bar{\beta}} \right)^{N_f} e^{-\bar{\beta}(D+m)} \right] = \\ &= \det \left[ e^{-\bar{\beta}(D+m)} \left( -i\gamma_5 \frac{d}{d\theta} \right)^{N_f} e^{i\bar{\theta}\gamma_5(D+m)} \right] \end{aligned} \quad (9)$$

which emphasizes the close explicit relation between fermionic and gauge contributions to the distribution density of full QCD.

Let us finally comment on three points.

( $\alpha$ ) Note that the gauge action density proportional to  $\text{tr}(D(U) - D(\mathbb{I}))_{n,n}$  is expected to be a non-ultralocal function of  $U$ . Thus, in coherent QCD we are necessarily giving up the possibility of strict reflection positivity at non-zero cutoff. However, this feature was abandoned already by requiring exact chiral symmetry of the fermionic action since such actions are necessarily non-ultralocal in fermionic variables [8]. Reflection positivity is expected to be recovered without problems in the continuum limit.

( $\beta$ ) Due to a complicated nature of Ginsparg–Wilson operators, it is expected that there exist configurations  $U$  for which  $D(U)$  is not uniquely defined. This can happen, for example, in fine-tuned backgrounds, where the number of zero modes (topology) changes. For such configurations the coherent gauge action is also not expected to be well-defined. We thus have to take these configurations out of the path integral (assign a zero distribution density to them) and operate under the assumption that we still obtain a universal theory (QCD) in the continuum limit. Again, however, we have made that choice already by using chiral fermions to begin with.

( $\gamma$ ) A noteworthy aspect of coherent LQCD is that the formulation puts the fermionic and gauge aspects of a numerical simulation with dynamical fermions on the same footing. The fact that fermionic and gauge parts of the action respond more coherently to proposed changes of the configuration in the Markov chain could be beneficial for algorithms that may be used in their simulation. Moreover, while adopting Ginsparg–Wilson fermions certainly makes the problem of dynamical simulations more involved, putting the gauge part of the action on par with fermions doesn't appear to generate qualitatively new complexities. Indeed, the simplest coherent LQCD discussed in this section requires simulating  $\det(D+m)^{N_f} e^{-\bar{\beta}D}$  instead of  $\det(D+m)^{N_f}$ , and this can be incorporated naturally in the existing algorithms. Detailed discussion of related issues will be given in Ref. [7].

### 3 More General Coherent LQCD

While the form of coherent LQCD introduced in Sec. 2 is perhaps the simplest one, there is in principle a much larger set of possibilities here. Indeed, by coherent LQCD we mean any formulation where the gauge and fermionic parts of the action are tied together in an explicit manner. Since the Dirac operator  $D$  defining fermionic action is a much more general (and yet highly constrained) object than the structure associated with the gauge action, it is natural to view  $D$  as primary and to model the gauge part accordingly.

Following this route, we can obtain a large class of coherent LQCD formulations if we treat general functions  $f(D)$  in a similar manner as we treated  $D$  in Sec. 2. In particular, if  $f(D)$  is a local operator and if  $\text{tr } f(D)_{n,n}$  is a scalar lattice field, then we generically expect the validity of a statement analogous to Conjecture C3 with equation (1) for smooth backgrounds replaced by

$$\text{tr} \left[ f(D(U(a)))_{0,0} - f(D(\mathbb{I}))_{0,0} \right] = -c^S a^4 \text{tr } F_{\mu\nu}(0) F_{\mu\nu}(0) + \mathcal{O}(a^6) \quad (10)$$

with the appropriate constant  $c^S$ . Proceeding in the same way also for the pseudoscalar case (i.e. considering operator  $\gamma_5 f(D)$ ) we obtain the definition of coherent LQCD in this case by replacing  $D$  in the gauge part of Eqs. (6,8) with  $f(D)$ , while Eq. (7) remains unchanged.

The freedom in choosing  $f(D)$  may allow us to formulate lattice QCD in a manner that casts the gauge and fermionic parts of the full action into an even more mutually symmetric form. To do that, let us consider the function  $f(D) = \ln(D + \eta)$  with fixed  $\eta > 0$ , and where the principal branch of the complex logarithm is used to define the operator. The analog of Eq. (6) is then (up to an inessential constant)

$$S_{\bar{\beta}, \bar{\theta}, \{m_f\}} = \text{Tr} (\bar{\beta} - i\bar{\theta}\gamma_5) \ln(D(U) + m_0) + \sum_{f=1}^{N_f} \bar{\psi}^f (D(U) + m_f) \psi^f \quad (11)$$

where we set  $\eta \equiv m_0 > 0$  to reflect its mass-like nature but still put it in contrast to quark masses  $m_f$ . Eq. (8) for distribution density with degenerate quark masses is replaced by

$$P_{\bar{\beta}, \bar{\theta}, m} \propto e^{\text{Tr} [N_f \ln(D+m) + (-\bar{\beta} + i\bar{\theta}\gamma_5) \ln(D+m_0)]} = \det \left[ (D + m)^{N_f} (D + m_0)^{-\bar{\beta} + i\bar{\theta}\gamma_5} \right] \quad (12)$$

where  $D \equiv D(U)$  and the formal relation  $A^B = e^{(\ln A)B}$  for square matrices  $A$  and  $B$  was used. We will refer to the above regularization as *logarithmic* LQCD for obvious reasons.

The validity of logarithmic LQCD relies on the following two conjectures (classical limit and locality) that we propose for the numerical and analytical investigation.

**Conjecture C4.** *Let  $A_\mu(x)$  be arbitrary smooth  $\text{su}(3)$  gauge potentials on  $\mathbb{R}^4$ . If  $U(a) \equiv \{U_{n,\mu}(a)\}$  is the transcription of this field to the hypercubic lattice with classical lattice spacing  $a$ , and  $\eta > 0$ , then equation (10) is valid for  $f(D) \equiv \ln(D + \eta)$  with generic  $D \in \mathcal{S}^F$  such that  $f(D)$  is well-defined. The non-zero constant  $c^S = c^S(\eta)$  is independent of  $A_\mu(x)$  at fixed  $D$ .*

**Conjecture C5.** *Let  $\eta > 0$  and  $D \in \mathcal{S}^F$  such that  $f(D) \equiv \ln(D + \eta)$  is well-defined. Then  $s_n(U) \equiv \text{tr } f(D(U))_{n,n}$  is a local composite of  $U$ .*

By saying that  $f(D)$  is “well-defined” in the above conjectures we mean that  $f(D(U))$  is a uniquely defined linear operator for all backgrounds  $U$  for which  $D(U)$  is uniquely defined. This is true e.g. for the family of overlap Dirac kernels  $D^{(\rho)}$ . It should be also mentioned that by *local* composite field  $s_n(U)$  we implicitly understand the strong version of locality condition which can be formulated as follows. Representing the configuration  $U$  in the canonical form used in Ref. [1], where  $U_{n,\mu} \leftrightarrow u(n,\mu) \equiv \{u_a(n,\mu), a = 1, 2, \dots, 8\}$  with  $u_a(n,\mu)$  being independent real parameters, locality of  $s_n(U)$  requires the existence of  $\alpha(\eta) > 0$  and  $A(\eta) \geq 0$  (independent of  $U$ ) such that

$$\max_{\mu,a} \left| \frac{\partial s_n(U)}{\partial u_a(m,\mu)} \right| \leq A(\eta) e^{-\alpha(\eta)|n-m|} \quad \forall n, m \quad (13)$$

and for all configurations  $U$  for which  $D(U)$  is uniquely defined.<sup>5</sup> It needs to be emphasized that locality in this form has not been proved rigorously even for simpler cases such as  $\text{tr} D_{n,n}^{(\rho)}$  with  $D^{(\rho)}$  being an overlap Dirac operator. However, one can at least support it by exploring the behavior numerically in some average sense or for a specific class of configurations as was done e.g. in Refs. [9, 10] for the case of fermionic locality. Note also that the coefficients  $\alpha$ ,  $A$  are implicitly assumed to be chosen in the “optimal way”, which is in principle obtained by fixing the maximal possible  $\alpha$  first, and then selecting the minimal  $A$  given that choice.

The heuristic argument for the validity of Conjecture C5 is as follows. Consider the asymptotically large value of  $\eta$ . In that case we have that  $\ln(D + \eta) = \ln \eta + \ln(1 + D/\eta) \approx \ln \eta + D/\eta$ . Thus for  $\eta \rightarrow \infty$  we expect exponential localization of  $\text{tr}[\ln(D + \eta)]_{n,n}$  with  $A(\eta) \rightarrow 0$  and  $\alpha(\eta) \rightarrow \alpha_D$ , where  $\alpha_D$  is the localization range of  $\text{tr} D_{n,n}$ . For finite  $\eta > 0$ , the operator  $\ln(1 + D/\eta) \equiv \sum_\lambda \psi_\lambda \ln(1 + \lambda/\eta) \psi_\lambda^\dagger$  (with  $D\psi_\lambda = \lambda\psi_\lambda$ ) is expected to be both well-defined and local since nothing non-analytic happens when performing a spectral transformation  $\lambda \rightarrow \ln(1 + \lambda/\eta)$  if the spectrum is bounded and if there are no negative real eigenvalues ( $\leq -\eta$ ). Both of these conditions are generically satisfied by elements of  $\mathcal{S}^F$ . For example, if we consider the overlap Dirac operator with  $\rho = 1$ , one would naively expect that something non-analytic could happen for  $\eta \leq 2$  because the expansion of  $\ln(1 + x)$  does not converge for  $|x| \geq 1$ . However, this is not the case (we do not rely on the expansion at all) since there are no singularities of the complex logarithm in the relevant part of the complex plane. The situation in this regard is similar to  $(D + \eta)^{-1} = \eta^{-1}(1 + D/\eta)^{-1}$  which is expected to be exponentially localized for arbitrary  $\eta > 0$  (exponential decay of the massive propagator) and an analogous argument applies.

We now wish to make the following remarks:

(i) Note that for fixed  $D \in \mathcal{S}^F$  the mass-like parameter  $m_0$  merely labels different formulations of coherent LQCD with  $f(D) = \ln(D + m_0)$ , and is fixed as the continuum limit is taken. The freedom in choosing  $m_0$  can be used to adjust the desired degree of lattice locality in gauge interactions.

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<sup>5</sup>Note that this definition implicitly assumes that we can exclude, without any harm, configurations for which  $D(U)$  is not uniquely defined. In addition, one might have to exclude a subset of configurations for which the condition (13) is not satisfied if it can be shown that these configurations will not contribute in the path integral. This makes both the rigor and the possible numerical verification of strong locality rather difficult to deal with if non-ultralocal operators are used.

(ii) We have implicitly assumed so far in our considerations that both parts of gauge interactions (scalar action and the  $\theta$ -term) are governed by the same function  $f(D)$ . This is natural from the point of view of “coherence” between different parts of the theory. However, it might prove useful in some circumstances to use different functions  $f^S(D)$  and  $f^P(D)$  for the scalar and pseudoscalar parts respectively. For example, in the case of logarithmic LQCD we would have

$$P_{\bar{\beta}, \bar{\theta}, m} \propto \det \left[ (D + m)^{N_f} (D + m_0^S)^{-\bar{\beta}} (D + m_0^P)^{i\bar{\theta}\gamma_5} \right] \quad (14)$$

instead of (12). We emphasize that the distributions in question are actually invariant under the change of  $f^P(D)$  (the global topological charge will not change), but the local behavior of the action will be different, which might be useful for certain theoretical considerations.

(iii) While the relation between the scalar constant  $c^S(D)$  corresponding to  $D$ , and  $c^S(f(D))$  is in general complicated, the pseudoscalar case can be very simple. For example, if  $D$  is an overlap operator with  $\rho = 1$ , then the pseudoscalar constant of  $\text{tr } \gamma_5 f(D)$  is given by

$$c^P(f(D)) = \frac{f(2) - f(0)}{2} c^P(D) \quad (15)$$

where we have assumed that  $[D, f(D)] = 0$ . The generalization to arbitrary  $D \in \mathcal{S}^F$  in terms of its real modes is straightforward. We emphasize that the strict topological nature of  $\text{tr } \gamma_5 D_{n,n}$  is inherited in the  $\text{tr } \gamma_5 f(D)_{n,n}$ .

(iv) Another choice of function  $f(D)$  that could be used to obtain coherent LQCD with tunable range of lattice locality is  $f(D) = (D + \eta)^{-1}$ . This possibility, while appearing impractical, has an interesting theoretical appeal since it involves an operator (quark propagator) which enters expressions for hadronic correlation functions.

(v) Conjecture C4 offers a straightforward insight into the nature of fermionic determinants for classical backgrounds. This is discussed in Appendix B.

(vi) It is amusing to note that, as can be seen from Eq. (12), the role of continuum limit driving parameter  $\bar{\beta}$  in logarithmic LQCD is analogous to that of the number of quark flavors. Indeed,  $\bar{\beta}$  can be viewed as counting the (continuous) number of additional fermions (if  $c^S(m_0) < 0$ ) or pseudofermions (if  $c^S(m_0) > 0$ ). The process of taking the continuum limit  $|\bar{\beta}| \rightarrow \infty$  involves engaging more and more of such particles. In the continuum limit there is infinitely many of them and they become infinitely heavy since  $m_0$  is kept fixed and thus  $m_0/a \rightarrow \infty$  as  $a \rightarrow 0$ .

(vii) Finally, we note that the straightforward generalization of Conjecture C3 is expected to be valid for arbitrary polynomials of  $D$ . For future reference (see Sec. 5), let us make the needed statement explicit.

**Conjecture C3g.** *Let  $A_\mu(x)$  be arbitrary smooth  $\text{su}(3)$  gauge potentials on  $\mathbb{R}^4$ . If  $U(a) \equiv \{U_{n,\mu}(a)\}$  is the transcription of this field to the hypercubic lattice with classical lattice spacing  $a$ , then equation (10) is valid for  $f(D) \equiv D^n$ , where  $n$  is a positive integer and  $D$  is a generic element of  $\mathcal{S}^F$ . The non-zero constant  $c^S$  is independent of  $A_\mu(x)$  at fixed  $D$ .*



## 4 Symmetric Logarithmic LQCD

Following the ideas outlined in the previous section, one can attempt to define coherent LQCD in the form which is even more symmetric with respect to fermions and bosons. This can be done by exploiting the expected behavior of functions  $c^S(\eta)$  and  $\alpha(\eta)$ ,  $A(\eta)$  associated with  $s_n \equiv \text{tr} \ln(D + \eta)_{n,n}$  in the  $\eta \rightarrow 0$  limit. It turns out that the resulting formulation that we present below is not expected to be local by the usual criteria of strong locality. However, we will argue that if the locality condition is weakened to the form which still appears physically acceptable (*weak locality*), our symmetric formulation could be considered local, and thus expected to define the correct continuum theory at least for some minimal number of quark flavors. Given the elegant appeal of this regularization and the insight it offers into an interplay between bosonic and fermionic degrees of freedom, we analyze these issues in some detail below.

Before we begin, let us recall that we have already concluded about the characteristics  $c^S(\eta)$ ,  $\alpha(\eta)$ ,  $A(\eta)$  of  $s_n$  that

$$\lim_{\eta \rightarrow \infty} c^S(\eta) = \lim_{\eta \rightarrow \infty} \frac{c_D^S}{\eta} = 0 \quad \lim_{\eta \rightarrow \infty} \alpha(\eta) = \alpha_D \quad \lim_{\eta \rightarrow \infty} A(\eta) = \lim_{\eta \rightarrow \infty} \frac{A_D}{\eta} = 0 \quad (16)$$

with  $c_D^S$ ,  $\alpha_D$ ,  $A_D$  denoting the corresponding constants for  $D \in \mathcal{S}^F$ . Also,  $c^S(\eta)$ ,  $\alpha(\eta)$  and  $A(\eta)$  are generically expected to be non-singular for  $\eta > 0$ . As for the behavior near the lattice massless limit  $\eta \rightarrow 0$ , one can make some very plausible guesses. We begin with discussion of  $c^S(\eta)$  in this regard, which will bring us directly to the symmetric formulation. Since the norm of  $\ln(D + \eta)$  will diverge logarithmically in the  $\eta \rightarrow 0$  limit for smooth configurations, it is expected that  $s_n$ , and thus  $c^S(\eta)$ , will diverge at least logarithmically as well. We thus propose the following conjecture to be verified via explicit calculations.

**Conjecture C6.** *Let  $D \in \mathcal{S}^F$  such that  $f(D) \equiv \ln(D + \eta)$  is well-defined for arbitrary  $\eta > 0$ . If  $c^S(\eta)$  is the associated classical coupling of  $s_n(U) \equiv \text{tr} f(D(U))_{n,n}$  to  $F_{\mu\nu}F_{\mu\nu}$  defined by Conjecture C4, then there exists  $\eta_0$  such that  $c^S(\eta)$  is monotonic for  $\eta \leq \eta_0$ , and  $\lim_{\eta \rightarrow 0} |c^S(\eta)| = \infty$ . Moreover, there exists a non-zero (possibly infinite) limit*

$$\lim_{\eta \rightarrow 0} \frac{c^S(\eta)}{\ln(\eta)} \equiv \lim_{\eta \rightarrow 0} \kappa^S(\eta) \equiv \kappa^S(0) \neq 0 \quad (17)$$

Note that the possibility  $\kappa^S(0) = \pm\infty$  is included here. With regard to Conjecture C6 it is worth mentioning that the anticipated divergence is manifestly present for the pseudoscalar constant  $c^P(\eta)$ . For the family of overlap Dirac operators  $D^{(\rho)}$  one can see easily that

$$c^P(\eta) = \frac{1}{16\pi^2} \ln\left(1 + \frac{2\rho}{\eta}\right) \implies \kappa^P(0) = -\frac{1}{16\pi^2} \quad (18)$$

### 4.1 Definition of the Lattice Action

The divergence in coupling of  $\text{tr} \ln(D + \eta)_{n,n}$  to  $F^2$  implied by Conjecture C6 suggests that we can trade the continuum-limit driving coupling  $\bar{\beta}$  of formulation (11,12) in favor of

“variable” gauge mass parameter  $m_0 = m_0(g)$ .<sup>6</sup> To see this explicitly, consider the action after integrating out fermions (effective action) of formulation (11,12) with  $\theta = 0$ .

$$-S_{m,\bar{\beta}}^{eff} = \text{Tr} [N_f \ln(D + m) - \bar{\beta} \ln(D + m_0)] \quad \bar{\beta} \equiv \frac{1}{2g^2 c^S(m_0)} \quad (19)$$

The correct classical limit at arbitrary  $g$  will be reproduced when we replace it with

$$-S_{m,m_0}^{eff} = \text{Tr} [N_f \ln(D + m) - \text{sgn}(c^S(m_0)) \ln(D + m_0)] \quad (20)$$

where  $\text{sgn}(x)$  denotes the sign function, and  $m_0 = m_0(g)$  is implicitly related to  $g$  via

$$|c^S(m_0)| \equiv \frac{1}{2g^2} \quad (21)$$

Note that the Conjecture C6 implies the existence of  $g_0 > 0$  such that  $g \in (0, g_0]$  is in one-to-one correspondence with  $m_0 \in (0, \eta_0]$ .

Let us now write this lattice regularization explicitly in its general form. One way to do that is to emphasize the fact that, instead of  $g, \theta, \{m_f\}$ , the parameters of QCD in our formulation come all naturally in the mass-like manner, namely  $m_0^S, m_0^P, \{m_f\}$ . Indeed, the action of *symmetric logarithmic* LQCD can be defined by (up to an inessential constant)

$$\begin{aligned} S_{m_0^S, m_0^P, \{m_f\}} &= \text{sgn}(c^S(m_0^S)) \text{Tr} \ln(D(U) + m_0^S) \\ &- i \text{sgn}(c^P(|m_0^P|)) \text{sgn}(m_0^P) \text{Tr} \gamma_5 \ln(D(U) + |m_0^P|) \\ &+ \sum_{f=1}^{N_f} \bar{\psi}^f (D(U) + m_f) \psi^f \end{aligned} \quad (22)$$

where  $g$  is related to  $m_0^S$  via Eq. (21), while  $\theta$  is related to  $m_0^P$  through

$$\theta = \text{sgn}(m_0^P) 16\pi^2 |c^P(|m_0^P|)| \quad \theta \in (-\pi, \pi] \quad (23)$$

Note that in the above definition we have allowed the pseudoscalar mass  $m_0^P$  to be negative so that the lattice action density preserves exactly the transformation property of the pseudoscalar part under  $\theta \rightarrow -\theta$ . We emphasize that  $\lim_{m_0^P \rightarrow \infty} c^P(m_0^P) = 0$ , and  $\theta = 0$  case is thus obtained in this limit (as well as in  $m_0^P \rightarrow -\infty$  limit). The above definition assumes that  $|c^P(\eta)|$  is a decreasing function of  $\eta$  in the range of positive values  $[m_0^{P,c}, \infty)$  such that  $\theta(m_0^{P,c}) \equiv \pi = 16\pi^2 |c^P(m_0^{P,c})|$ . This is true for the family of overlap Dirac operators  $D^{(\rho)}$ , in which case  $c^P(m_0^P)$  is given by Eq. (18) and we have

$$m_0^{P,c} = \frac{2\rho}{e^\pi - 1} \quad (24)$$

The mass-like parameters in the above lattice theory are thus chosen to vary in the ranges

$$m_f \in (0, \infty) \quad m_0^S \in (0, m_0^{S,c}] \quad m_0^P \in [m_0^{P,c}, \infty) \cup (-\infty, -m_0^{P,c}) \quad (25)$$

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<sup>6</sup>We thank Andrei Alexandru for sparking this line of thought in our conversations.

where  $m_0^{S,c}$  is the maximal  $\mu_0$  satisfying the statement of Conjecture C7. The continuum limit with  $N_f \leq 16$  is taken via  $m_0^S \equiv m_0^S(a) \rightarrow 0$  while decreasing  $m_f = m_f(a) \propto a\bar{m}_f^r$  towards zero so that some set of renormalized masses  $\bar{m}_f^r$  (in physical units) is held fixed, and with the pseudoscalar mass  $m_0^P$  (specifying  $\theta$ ) kept constant in the process.<sup>7</sup>

The second possibility of a general definition that is worth mentioning is obtained by setting  $m_0^S = m_0^P \equiv m_0$  with the scalar part of the gauge action being identical to that of (22) (with continuum limit driven by changing  $m_0$  according to (21)), but with the pseudoscalar part being constructed as in logarithmic LQCD (11,12). The virtue of such definition is that both scalar and pseudoscalar action densities then involve gauge operators with the same locality properties driven by  $m_0$ . This is relevant if one is interested in studying the space–time behavior of the action. We emphasize again that the actual distribution density  $P(U)$  (and thus the typical configurations) is invariant under various choices of  $\theta$ –term constructed from fixed  $D \in \mathcal{S}^f$ . Indeed, all such choices yield by construction the same global topological charge  $Q(U)$  as the one defined by  $D(U)$ . The different formulations here will differ only by the space–time behavior of the pseudoscalar (topological) action density.

## 4.2 The Locality Issue

Let us now turn to the expected behavior of  $\alpha(\eta)$  for  $\eta \rightarrow 0$ . Here one expects that  $\lim_{\eta \rightarrow 0} \alpha(\eta) = 0$  since the usual view is that the effective action of massless fermion is non–local. One of the arguments supporting this conclusion is that the fermion operator  $\ln(D + \eta)$  in free background  $U \equiv \mathbb{I}$  is non–local in fermionic variables in that limit, i.e.  $\|\ln(D(\mathbb{I}) + \eta)_{n,m}\| \propto \exp(-\gamma(\eta)|n - m|)$  with  $\lim_{\eta \rightarrow 0} \gamma(\eta) = 0$ . Indeed, for sufficiently small  $\eta$ , the singularity of the operator in the Fourier space closest to real momenta is distance  $\eta$  away, and hence this is also an inverse range of the operator. While it is not obvious a priori that the locality in fermionic and gauge variables have to be strictly related, this is expected to be generically true. This is expressed in the following conjecture

**Conjecture C7.** *Let  $D \in \mathcal{S}^F$  such that  $f(D) \equiv \ln(D + \eta)$  is well–defined for arbitrary  $\eta > 0$ . If  $\alpha(\eta)$  is the inverse range of  $s_n(U) \equiv \text{tr } f(D(U))_{n,n}$ , then  $\lim_{\eta \rightarrow 0} \alpha(\eta) = 0$ .*

Immediate issue with symmetric logarithmic LQCD (22) is that, assuming the validity of Conjecture C7, it is non–local by the standard definition of (strong) locality used so far in this discussion (see Eq. (13)). Indeed, let us analyze this in more detail. Writing the lattice gauge action in the standard form  $S^G(U) = \beta \sum_n O_n(U, g)$  (see [1]) the requirement of locality is that the operator  $O_n(U, g)$  satisfies Eq. (13) independently of  $g$ , at least for sufficiently small  $g$ . In other words, even though the optimal locality parameters  $A(g)$ ,  $\alpha(g)$  can depend on  $g$ , we can select constants  $A \geq 0$  and  $\alpha > 0$  such that (13) is satisfied irrespectively of  $g$ . If  $A(g)$ ,  $\alpha(g)$  are continuous functions for  $g > 0$ , then this happens if  $0 \leq \lim_{g \rightarrow 0} A(g) < \infty$  and  $0 < \lim_{g \rightarrow 0} \alpha(g)$ . However, in our case we have ( $m_0 \equiv m_0^S$ )

$$O_n = \frac{1}{12 c^S(m_0)} \text{tr } \ln(D + m_0)_{n,n} = \frac{1}{12 c^S(m_0)} s_n \quad (26)$$

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<sup>7</sup>Note that throughout the paper we could also use the definition of the massive Dirac operator which also scales  $D$  such that the spectrum is guaranteed to be within the fixed compact region. In case of overlap Dirac operator this would mean to consider  $(1 - \eta/2\rho)D^{(\rho)} + \eta$  instead of  $D^{(\rho)} + \eta$ . Such definition would yield all the ranges in (25) to be finite intervals.

which, according to Conjecture C7, is expected to be non-local in this strong sense because  $\lim_{g \rightarrow 0} \alpha(m_0(g)) = 0$  for  $s_n$ .

While strong locality is believed to be sufficient for the universality of the continuum limit, it is not clear that it is also necessary. Moreover, for non-ultralocal operators it is hard to guarantee the condition for all configurations, and one is relegated to possibly excluding a certain subset of them that will not satisfy it, but are not believed to be contributing in the continuum limit. However, the relevance in the continuum limit can in principle depend on the number of flavors, and possibly also on which lattice regularization is used to define it, which makes related arguments very hard to make rigorous. The above two points suggest that it is perhaps sensible to revise the requirements for locality of the theory in two ways. (1) Require only that the range of locality scales to zero in *physical units* as the continuum limit is approached. (2) To view the notion of locality for an operator only in conjunction with the theory (sequence of distribution densities) in which it is used, i.e. to assign the locality property to a pair (*lattice operator, lattice theory*) rather than to the operator alone.

Let us now outline such definition of locality for the case of scalar (or pseudoscalar) operators that only depend on gauge fields, since such operators are relevant for our discussion. According to the comments made above, the starting point in the definition involves fixing the lattice “theory”  $T$  considered. This, in turn, means choosing action  $S(p)$  depending on the set of bare lattice parameters  $p$  together with the prescription of assigning the lattice spacing  $a$  to the model, as well as a prescription for changing the bare parameters  $p = p(a, \bar{p})$  for the continuum limit  $a \rightarrow 0$  to be taken with some set of external physical parameters  $\bar{p}$  being fixed. For example, in QCD with two degenerate flavors of quarks with  $\theta = 0$ , if we use the lattice action (22), then  $p \equiv \{m_0^S, m\}$ , the lattice spacing can be assigned e.g. by fixing the lowest pseudoscalar glueball mass, and  $\bar{p} \equiv \{\bar{m}^T\}$  can be a fixed renormalized quark mass in some particular scheme. We can thus schematically write specification of the lattice theory as  $T(\bar{p}) \equiv \{S(p), a, p(a, \bar{p})\}$ .

Consider an arbitrary scalar (pseudoscalar) operator  $O_n = O_n(U, p)$ , where we allowed for a possible dependence of  $O$  on the same bare parameters as the theory in which its locality will be considered. For arbitrary configuration  $U$  let us define the function  $\Delta_{n,m}^O(U, p)$  via (see Eq. (13))

$$\Delta_{n,m}^O(U, p) \equiv \max_{\mu, b} \left| \frac{\partial O_n(U, p)}{\partial u_b(m, \mu)} \right| \quad (27)$$

and its average in the ensemble corresponding to theory  $T(\bar{p})$  (with  $\bar{p}$  being fixed and not explicitly denoted) at given lattice spacing  $a$  as

$$\Delta_{n,m}^O(T, a) \equiv \left\langle \Delta_{n,m}^O(U, p(a)) \right\rangle_{S(p(a))} \quad (28)$$

where  $\langle \cdot \rangle_{S(p(a))}$  denotes ensemble average over the distribution specified by  $S(U, \psi, \bar{\psi}; p(a))$  in the usual sense. We then consider the operator  $O$  being *weakly local* relative to the theory  $T(\bar{p})$  if the following two conditions are satisfied.

- (i) There is a lattice spacing  $a_0 > 0$  such that for all  $0 < a < a_0$  there exist  $A^T(a) \geq 0$  and  $\alpha^T(a) > 0$  such that

$$\Delta_{n,m}^O(T, a) \leq A^T(a) e^{-\alpha^T(a) |n-m|} \quad \forall n, m \quad (29)$$

As before, it is implicitly assumed here that  $A^T(a)$ ,  $\alpha^T(a)$  are chosen in an “optimal” way, i.e. maximal  $\alpha^T(a)$  and minimal  $A^T(a)$ .

- (ii) The coupling to gauge fields at arbitrary non-zero physical distance  $\bar{r} \equiv |n - m|a$  vanishes in the continuum limit both in absolute terms, and relative to coupling at  $\bar{r} = 0$ , i.e.

$$\lim_{a \rightarrow 0} A^T(a) e^{-\alpha^T(a) \bar{r}/a} = \lim_{a \rightarrow 0} e^{-\alpha^T(a) \bar{r}/a} = 0 \quad \forall \bar{r} > 0 \quad (30)$$

Note that condition (ii) in fact involves two requirements. In the usual situation when  $\lim_{a \rightarrow 0} A^T(a) = A^T(0) < \infty$ , this translates into a single requirement that the range of the interaction defined by  $\alpha^T(a)$  goes to zero in physical units. In other words

$$\lim_{a \rightarrow 0} a \frac{1}{\alpha^T(a)} = 0 \quad (31)$$

In the next section we discuss the possibility that symmetric logarithmic LQCD is weakly local.

### 4.3 Possibilities for Weak Locality

With the acceptance of non-ultralocal operators (the conceptual leap taken decisively by using an overlap operator), the aspect of locality acquired a much more prominent role in lattice QCD than it used to. While with ultralocal operators locality was a non-issue decided before any investigations of dynamical behavior of the theory began, with non-ultralocal operators it appears that locality needs to be viewed in the dynamical context and, as such, is to be determined a posteriori. In this situation there exists a danger that one will either mistakenly put his faith in the formulation which will eventually turn out not to define QCD, or that one will mistakenly discard formulations with beautiful properties assuming improperly that they are non-local. It is thus important that one has a sensible “guessing guide” to make reasonable choices. It is in this context that we view the possible usefulness of *weak locality*. In other words, we propose that if lattice formulation has a sensible chance to satisfy weak locality, then it also has a sensible chance to define QCD in the continuum limit.

In this section we suggest that the possibility of symmetric logarithmic LQCD (22) being weakly local depends crucially on the nature of expected divergence of  $c^S(\eta)$  in the  $\eta \rightarrow 0$  limit. To do that, let us assume that we wish to define QCD with  $N_f$  flavors of quarks via symmetric logarithmic LQCD, and we will set  $\theta = 0$  for simplicity. We thus have that  $p \equiv \{m_0^S \equiv m_0, m_1, m_2, \dots, m_{N_f}\}$  and, to define a theory  $T$ , we fix the procedure for determining the lattice spacing, as well as a scheme in which renormalized quark masses  $\bar{p} \equiv \{\bar{m}_1^r, \bar{m}_2^r, \dots, \bar{m}_{N_f}^r\}$  are being fixed in physical units as the continuum limit is taken. Conjecture C5 implies the existence of coefficients  $A^T(a)$ ,  $\alpha^T(a)$  characterizing the operator  $s_n(U, m_0(a)) \equiv \text{tr} \ln(D(U) + m_0(a))_{n,n}$  along the path to the continuum limit in this theory. We are interested in the locality properties of the gauge action  $O_n(U, m_0(a)) \equiv s_n(U, m_0(a))/12c^S(m_0(a))$  (see Eq. (26)). For arbitrary  $\bar{p}$  it is expected that  $A^T(a)/c^S(m_0(a))$  does not diverge for  $a \rightarrow 0$  since the logarithmic divergence in  $s_n$

for  $m_0 \rightarrow 0$  is removed away in  $O_n$  via division by  $c^S(m_0)$ .<sup>8</sup> At the same time the inverse ranges of locality for  $s_n$  and  $O_n$  are identical. Consequently, the validity of weak locality for gauge action of symmetric logarithmic QCD translates into validity of condition (31) or, emphasizing the implicit  $\bar{p}$  dependence of the whole procedure

$$\lim_{a \rightarrow 0} \frac{a(\bar{p})}{\alpha^T(m_0(a(\bar{p})))} = 0 \quad (32)$$

For asymptotically free theory ( $N_f \leq 16$ ), the continuum limit  $a \rightarrow 0$  is realized via  $m_0 \rightarrow 0$  due to Conjecture C6 and Eq. (21). In this regime, the localization range of  $s_n$  in the ensemble average is expected to be crucially driven by  $m_0$  irrespectively of the theory we are following (see the discussion introducing the Conjecture C7 as well as Conjecture C8 in Appendix C). In particular,  $\alpha^T(m_0) \propto m_0$  for  $m_0 \rightarrow 0$ . Consequently, for sufficiently small lattice spacings, where the asymptotic 1-loop scaling formula relating  $a$  to bare coupling  $g$  (and hence  $m_0$ ) can be used, we require that

$$0 = \lim_{m_0 \rightarrow 0} \frac{\exp(-\frac{|c^S(m_0)|}{\beta_0})}{m_0} = \lim_{m_0 \rightarrow 0} m_0^{\frac{|\kappa^S(0)|}{\beta_0} - 1} \quad (33)$$

where  $\beta_0 = (11 - \frac{2}{3}N_f)/16\pi^2$ , and the relation (21) as well as Conjecture C6 were used.

Equation (33) implies that symmetric logarithmic LQCD for  $N_f$  asymptotically free flavors ( $N_f \leq 16$ ) is expected to be weakly local only if

$$\frac{|\kappa^S(0)|}{\beta_0} > 1 \quad (34)$$

Since  $\beta_0 > 0$  for  $N_f \leq 16$ , this means that  $|\kappa^S(0)| > \beta_0$  which can, in turn, be viewed as a restriction on the number of asymptotically free flavors for which the definition via symmetric logarithmic LQCD is possible. In particular,

$$\frac{33}{2} - 24\pi^2 |\kappa^S(0)| < N_f \quad (35)$$

We have thus arrived at a rather intriguing conclusion, namely that the suitability of  $\text{Tr} \ln(D + m_0)$  to be a gauge action with  $m_0$  controlling the continuum limit can in principle depend on the number of flavors in the theory we wish to define. In particular, there is possibly a minimal number of flavors for which asymptotically free SU(3) gauge theory can be defined in this symmetric manner. For example, pure glue QCD ( $N_f = 0$ ) can only be formulated if

$$|\kappa^S(0)| > \frac{33}{48\pi^2} \quad (36)$$

It is not possible to define any asymptotically free SU(3) gauge theory via symmetric logarithmic LQCD if  $|\kappa^S(0)| \leq 1/48\pi^2$ . The fact that the consistency of this maximally symmetric coherent LQCD might dictate the minimal number of quark flavors is an extreme example

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<sup>8</sup>The heuristic argument applies for smooth configurations but is not expected to be violated in the ensemble average. See Conjecture C8 in Appendix C.

of how strongly the unified description of gauge and fermionic aspects of the theory can interrelate the two. Needless to say, computing  $\kappa^S(\mu)$  for the family of overlap Dirac operators  $D^{(\rho)}$  would shed a rather intriguing detail on this point, and this will be pursued.

Finally, let us remark that in derivation of result (35) we have used, apart from Conjectures C4–C7, two other ingredients that are expected to manifestly hold, but are not proved. We give the relevant statements in the Appendix C and propose to examine their validity numerically for the case of overlap Dirac operator.

## 4.4 Discussion

To highlight the appeal of symmetric logarithmic LQCD, let us discuss in more detail how contributions of quarks and gluons to the total distribution density of gauge configurations appear in the completely form-symmetric manner. Consider the theory at  $\theta = 0$  with its corresponding effective action and the associated distribution density, namely

$$\begin{aligned} -S_{\{m_f\}}^{eff} &= \text{Tr} \sum_{f=0}^{N_f} \ln \left( D(U) + m_f \right) = \text{Tr} \ln \prod_{f=0}^{N_f} \left( D(U) + m_f \right) \\ P_{\{m_f\}} &\propto \exp \left( -S^{eff}(U) \right) = \det \prod_{f=0}^{N_f} \left( D(U) + m_f \right) \end{aligned} \quad (37)$$

where we used Eq. (22) and it was implicitly assumed that  $c^S(m_0) < 0$  for sufficiently small  $m_0$ .<sup>9</sup> Thus, in this regularization, the gauge field contribution to the total action is equivalent to adding an additional flavor of quarks. Indeed, one can write the total action of symmetric logarithmic LQCD in the form

$$S_{\{m_f\}} = \sum_{f=0}^{N_f} \bar{\psi}^f \left( D(U) + m_f \right) \psi^f \quad (38)$$

where the additional flavor  $\psi^0, \bar{\psi}^0$  becomes infinitely heavy (in physical units) in the continuum limit, and decouples from the light “physical” flavors in arbitrary fermionic correlation functions. Indeed, sufficiently close to the continuum limit we have

$$\lim_{a \rightarrow 0} \frac{m_0(a)}{a} \propto \lim_{m_0 \rightarrow 0} m_0^{1 - \frac{|\kappa^S(0)|}{\beta_0}} = \infty \quad (39)$$

if Eq. (33) guaranteeing weak locality is satisfied.

It is also instructive to emphasize the dual view, where the same lattice dynamics (at the effective action level) can be considered as being defined by the collection of  $N_f + 1$  form-symmetric gauge action terms controlled by  $N_f + 1$  distinct coupling constants. Indeed, the lattice action density can be written in the form

$$S_n^{eff} = \sum_{f=0}^{N_f} -\frac{1}{2g_f^2} F_n^2(U; g_f) \quad (40)$$

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<sup>9</sup>One can make very heuristic arguments for this to be the correct sign. Preliminary numerical results confirm this expectation at least for the overlap Dirac operator [6].

where the (scalar) lattice operator  $F^2(U; g)$  is defined via

$$F_n^2(U; g) = 2g^2 \operatorname{tr} \left[ \ln \left( D(U) + \eta(g) \right) \right]_{n,n} \quad |c^S(\eta)| \equiv \frac{1}{2g^2} \quad (41)$$

We note again that we assumed that  $c^S(m_0) < 0$  for sufficiently small  $m_0$  and thus, strictly speaking, the above lattice formulation has the proper classical limit only for sufficiently small  $g_0$ . Among other things, form (40) nicely illustrates the point that locality should be viewed as a “theory-dependent” concept. Indeed, while the same operator  $F^2$  is used to define the full theory, only the term driven by  $g_0$  (gauge action) involves weakly local operator along the path to the continuum limit. Indeed, for arbitrary set of fixed renormalized masses  $\bar{p} \equiv \{\bar{m}_1^r, \bar{m}_2^r, \dots, \bar{m}_{N_f}^r\}$ , the term driven by coupling constant  $g_f$  for  $f > 0$ , will have the range proportional to  $1/\bar{m}_f^r$  in the continuum limit, and is manifestly non-local.

## 5 Classically Coherent LQCD

While our rationale for proposing coherent LQCD was based entirely on the considerations of QCD vacuum structure developed in Ref. [1], there is an additional and seemingly unrelated motivation for it. In formulating lattice regularizations we are almost exclusively guided by formal equations in the continuum. Indeed, we require that relevant expressions are reproduced in the classical limit. Moreover, we try to arrange that symmetries of the continuum theory are respected by lattice dynamics to the largest extent possible. This is believed to ensure universality and also make the transition to the continuum limit smoother. However, there is another possible element in such continuum–lattice correspondence that is usually not taken into account. To see this, let us rewrite the continuum expression for QCD action (59,60) in a different manner. In particular, sorting out different Clifford components of  $D^2 \equiv (D_\mu \times \gamma_\mu)^2$  and  $D^4$  one can easily check that

$$\left( D^2 - D_\mu D_\mu \times \mathbb{I}^s \right)^2 \psi(x) = \left( -\frac{1}{2} F_{\mu\nu}(x) F_{\mu\nu}(x) \times \mathbb{I}^s + \frac{1}{2} F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x) \times \gamma_5 \right) \psi(x) \quad (42)$$

and

$$\left( D^4 - (D_\mu D_\mu)^2 \times \mathbb{I}^s \right) \psi(x) = \left( -\frac{1}{2} F_{\mu\nu}(x) F_{\mu\nu}(x) \times \mathbb{I}^s + \frac{1}{2} F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x) \times \gamma_5 + \dots \right) \psi(x) \quad (43)$$

One can thus formally write the continuum action as

$$\begin{aligned} S &= \operatorname{Tr} \left( \frac{1}{4g^2} + i \frac{\theta}{32\pi^2} \gamma_5 \right) \left( D^2 - D_\mu D_\mu \times \mathbb{I}^s \right)^2 + \bar{\psi} (D + m) \psi \\ &= \operatorname{Tr} \left( \frac{1}{4g^2} + i \frac{\theta}{32\pi^2} \gamma_5 \right) \left( D^4 - (D_\mu D_\mu)^2 \times \mathbb{I}^s \right) + \bar{\psi} (D + m) \psi \end{aligned} \quad (44)$$

Note that the above expression makes explicit the observation that we promoted in this article starting from very different motivation (principle of chiral ordering). In particular, it can be interpreted as suggesting that the fundamental object for formulation of QCD is massless Dirac operator  $D$ , and that gauge and fermionic aspects of the theory are explicitly connected to one another because of that.



We emphasize that the above motivation for coherent LQCD is classical in nature since the corresponding formal equations in the continuum can be made meaningful for fields that are smooth almost everywhere. These equations suggest that from a classical point of view, a natural way to proceed with definition of coherent LQCD is to choose arbitrary lattice Dirac operator  $D \in \mathcal{S}^F$ , and then form the coherent LQCD via the prescription given in Secs. 2 and 3, with  $f(D) = D^4$ . We will refer to this formulation as *classically coherent* LQCD.<sup>10</sup> Note that the statement on classical limit needed for this formulation is incorporated in Conjecture C3g, and the locality follows from locality of  $D$ .

We wish to make two points here.

(i) Consider the set  $\bar{\mathcal{S}}^F \supset \mathcal{S}^F$  of lattice Dirac operators  $D$  that satisfy all conditions specifying  $\mathcal{S}^F$  except the condition of lattice chiral symmetry. Then we can still follow the procedure above, and associate with arbitrary  $D \in \bar{\mathcal{S}}^F$  the scalar and pseudoscalar densities, as well as the gauge action via traces of  $f(D) = D^4$ . We will refer to these gauge objects as classically associated with  $D$ . Thus, for example, the scalar gauge action classically associated with Wilson fermions is a Wilson gauge action. The use of Wilson gauge action in combination with overlap fermions appears highly incoherent from this point of view. Needless to say, it would be of interest to study the properties of the gauge action classically associated with overlap fermions.

(ii) It should be emphasized here that we do not necessarily view classically coherent LQCD as being privileged over other choices. Rather, our view is that, at least for issues related to QCD vacuum structure, the selection of coherent formulation that makes the transition to the continuum limit smooth will depend on the nature of configurations dominating the QCD path integral. This is obviously an open issue.

## 6 Other Operators

Apart from large freedom in choosing the action for lattice regularization, there is an analogous freedom in choosing other lattice operators for measuring physical observables. In line with the point of view taken in this paper, we would like to use operators that are explicit functions of lattice Dirac kernel  $D$ . This would bring the coherence (in the sense talked about here) to the entire process of extracting QCD predictions via lattice definition. It would also most likely lead to a high degree of space-time order in typical configurations of gauge-invariant composite fields. In fact, using the topological charge density constructed this way led to the basic finding that fundamental topological structure in the QCD vacuum exists [11, 12]. Before we start, we should point out that all fermionic observables (or fermionic parts of mixed observables) are already expressed coherently since they are functions of  $(D + m)^{-1}$ , and hence  $D$ . Thus, our aim is basically to do the same for gauge observables.

The generic way of achieving this is to use *chiral ordering* transformations of the gauge field defined in Ref. [1]. Indeed, to every operator  $O(U)$  of interest, we can assign a related

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<sup>10</sup>One can also define coherent LQCD that strictly follows the form of Eq. (44). This will be discussed in required detail elsewhere.

operator

$$O^{\mathcal{M}^D}(U) \equiv O(\mathcal{M}^D(U)) \quad (45)$$

where  $\mathcal{M}^D$  is the chiral ordering transformation associated with operator  $D$ . Transformation  $\mathcal{M}^D$  extracts the effective  $SU(3)$  phase acquired by chiral fermion when hopping from  $n + \mu$  to  $n$  relative to the free case. Simple version of such transformation discussed in [1] is given by  $\mathcal{M}^D = \mathcal{M}^{(3)} \circ \mathcal{M}^{(2)} \circ \mathcal{M}^{(1)}$  with

$$\mathcal{M}_{n,\mu}^{(1)}(U) = \frac{1}{4} \text{tr}^s \left[ (D_{n,n+\mu}(\mathbb{I}))^{-1} D_{n,n+\mu}(U) \right] \quad (46)$$

Maps  $\mathcal{M}^{(2)}$  and  $\mathcal{M}^{(3)}$  represent the unitary and group projections respectively. Clearly, the operator  $O^{\mathcal{M}^D}(U)$  depends explicitly on  $D$ . We emphasize that for any local gauge-invariant operator  $O$ , the associated operator  $O^{\mathcal{M}^D}$  inherits its transformation properties, locality, and its classical limit. Similarly, other observables of interest, such as large Wilson loops, will inherit their required properties.

While the above construction achieves the goal of complete coherence for all aspects of the theory, it might be useful to have explicit representation for certain relevant operators in the same way we have for scalar density and pseudoscalar density. To do that, let us decompose the lattice Dirac operator  $D \in \mathcal{S}^F$  in its Clifford components, namely

$$D = \sum_{i=1}^{16} \mathcal{G}^i \times \Gamma^i \quad \mathcal{G}_{n,m}^i = \frac{1}{4} \text{tr}^s \Gamma^i D_{n,m} \quad (47)$$

where  $\Gamma \equiv \{\Gamma^i, i = 1, \dots, 16\}$  is the complete orthogonal Clifford basis such that  $(\Gamma^i)^2 = \mathbb{I}^s$ . We will use  $\Gamma \equiv \{\mathbb{I}^s, \gamma_\mu, \sigma_{\mu\nu} (\mu < \nu), i\gamma_\mu \gamma_5, \gamma_5\}$ , where  $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ ,  $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ , and  $\gamma$ -matrices satisfy  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu} \mathbb{I}^s$ . We then write the Clifford decomposition in the form

$$D = \mathcal{S} \times \mathbb{I} + \mathcal{V}_\mu \times \gamma_\mu + \mathcal{T}_{\mu\nu} \times \sigma_{\mu\nu} + \mathcal{A}_\mu \times (i\gamma_\mu \gamma_5) + \mathcal{P} \times \gamma_5 \quad (48)$$

where the gauge covariant operators  $\{\mathcal{S}, \mathcal{V}_\mu, \mathcal{T}_{\mu\nu}, \mathcal{A}_\mu, \mathcal{P}\}$  carry space-color indices, and have well-defined transformation properties under the hypercubic group. If  $\gamma$ -matrices are Hermitian (which we assume in what follows) then  $\mathcal{S}, \mathcal{T}_{\mu\nu}, \mathcal{P}$  are also Hermitian while  $\mathcal{V}_\mu, \mathcal{A}_\mu$  are anti-Hermitian due to  $\gamma_5$ -Hermiticity of  $D$ . While there are several possibilities for extracting useful gauge operators from the above Clifford components, we focus here on local operators obtained from  $D_{n,n}$ . This work is already based on the fact that useful operators of this type can be obtained from  $\mathcal{S}$  and  $\mathcal{P}$ , namely that (see Conjecture C3)

$$4 \text{tr} \mathcal{S}(U(a))_{0,0} = C - c^S a^4 \text{tr} F_{\mu\nu}(0) F_{\mu\nu}(0) + \mathcal{O}(a^6) \quad (49)$$

and (see Eq. (3))

$$4 \text{tr} \mathcal{P}(U(a))_{0,0} = -c^P a^4 \text{tr} F_{\mu\nu}(0) \tilde{F}_{\mu\nu}(0) + \mathcal{O}(a^6) \quad (50)$$

where  $C$  is a constant and  $U(a)$  is a gauge configuration obtained from arbitrary smooth gauge potentials  $A_\mu(x)$  at classical lattice spacing  $a$  using prescription (2). To obtain the predictions for classical limits of other Clifford components, we use the same logic that is

behind the above results. In particular, we try to identify the most general gauge covariant continuum operators of given engineering dimension with definite transformation properties under hypercubic subgroup of  $O(4)$ . Such procedure proceeds by first finding the most general combinations of monomials in covariant derivative  $D_\mu$  at given degree (engineering dimension), that do not involve any derivatives of the vector acted upon. The result of this straightforward manipulation obviously yields a constant for dimension 0, and that there are no operators of dimension 1. Moreover, linear combinations of  $F_{\mu\nu}$  are the only possibility for dimension 2, while linear combinations of  $[F_{\mu\nu}, D_\rho] = [F_{\mu\nu}, A_\rho] - \partial_\rho F_{\mu\nu}$  are the only possibility for dimension 3. This information is sufficient to deduce the leading terms in the classical expansion for  $\mathcal{V}_\mu$ ,  $\mathcal{T}_{\mu\nu}$  and  $\mathcal{A}_\mu$ . In particular,

$$4 \mathcal{V}_\mu(U(a))_{0,0} = -c^V a^3 ([F_{\mu\nu}, A_\nu] - \partial_\nu F_{\mu\nu})_{x=0} + \mathcal{O}(a^5) \quad (51)$$

$$4 \mathcal{T}_{\mu\nu}(U(a))_{0,0} = i c^T a^2 F_{\mu\nu}(0) + \mathcal{O}(a^4) \quad (52)$$

$$4 \mathcal{A}_\mu(U(a))_{0,0} = -c^A a^3 \epsilon_{\mu\nu\rho\sigma} ([F_{\nu\rho}, A_\sigma] - \partial_\sigma F_{\nu\rho})_{x=0} + \mathcal{O}(a^5) \quad (53)$$

Thus, while the operators associated with  $\mathcal{V}_\mu$ ,  $\mathcal{A}_\mu$  do not appear particularly significant, the field-strength tensor is certainly relevant for many applications including the study of QCD vacuum structure. We point out that the relation equivalent to (52) has previously been mentioned in Ref. [13]. It needs to be emphasized that the above arguments leading to Eqs. (51–53) do not constitute the proof of their validity. Indeed, they rather suggest the conjectures analogous to Conjecture C3. Below we state this explicitly for the relevant case of  $\mathcal{T}_{\mu\nu}$ . Its validity for overlap Dirac operator will be examined in Refs. [6, 14].

**Conjecture C9.** *Let  $A_\mu(x)$  be arbitrary smooth  $\text{su}(3)$  gauge potentials on  $\mathbb{R}^4$ . If  $U(a)$  is the transcription of this field to the hypercubic lattice with classical lattice spacing  $a$  then*

$$\text{tr}^s \sigma_{\mu\nu} D_{0,0}(U(a)) = i c^T a^2 F_{\mu\nu}(0) + \mathcal{O}(a^4) \quad (54)$$

for generic  $D \in \mathcal{S}^F$ . Here  $c^T$  is a non-zero constant independent of  $A_\mu(x)$  at fixed  $D$ .

Finally, we point out that the conclusions on classical limits for Clifford components of  $D \in \mathcal{S}^F$  described here are also expected to be valid for more general operators  $f(D)$ , including all the cases discussed in this manuscript.

## 7 Effective LQCD II.

Considerations on coherent LQCD offer a different viewpoint on the notion of *effective* LQCD at given fermionic response scale  $\Lambda_F$ . The first form of effective LQCD, described in Ref. [1], is based on the eigenmode expansion of chirally ordered gauge field.<sup>11</sup> The basic logic of the construction is as follows. Starting from some original theory  $S(a)$  at lattice spacing  $a$ , we consider the theory  $S^{\mathcal{M}^D}(\bar{a})$  obtained by transforming the ensemble of  $S(a)$  via chiral ordering transformation  $\mathcal{M}^D$  (such as one based on Eq. (46)). Here  $D \in \mathcal{S}^F$  is the Dirac

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<sup>11</sup>The construction in [1] has its roots in Ref. [15, 16], and bears some technical similarities to “Laplacian filtering” proposed in Ref. [17] (see also [18]).

operator used in  $S$ , and  $\bar{a}(a) \approx a$ . The ensemble corresponding to effective theory  $S_{\Lambda_F}^{\mathcal{M}^D}$  is then obtained by performing this transformation with  $\mathcal{M}^{D, \bar{a}\Lambda_F}$ , based on  $D^{\bar{a}\Lambda_F}$  rather than  $D$ . Here  $D^{\lambda_F}$  represents an eigenmode expansion of  $D$  including eigenvalues with magnitudes up to  $\lambda_F$  in lattice units. Upon taking the continuum limit, the theory defined by  $S_{\Lambda_F}^{\mathcal{M}^D}$  is naively expected to be described by non-local interaction with the range of locality related to  $1/\Lambda_F$ , but the lattice action itself is only known implicitly via its numerical ensemble.<sup>12</sup>

If one starts with coherent LQCD, then a natural possibility opens itself immediately for *explicit* definition of a non-local theory that could play an analogous role. Indeed, consider the action (6) for simplest version of coherent LQCD. At  $\theta = 0$  and with  $N_f$  degenerate quark flavors we can write the corresponding probability distribution of gauge fields as

$$P \propto e^{\text{Tr}[N_f \ln(D+m) - \bar{\beta}D]} = \det \left[ (D+m)^{N_f} e^{-\bar{\beta}D} \right] = \prod_{\lambda} (\lambda+m)^{N_f} e^{-\bar{\beta}\lambda} \quad (55)$$

If the “mother theory” is at lattice spacing  $a$ , then we define the associated effective LQCD at fermionic response scale  $\Lambda_F$  via

$$P_{\Lambda_F}(a) \propto \prod_{|\lambda| \leq a\Lambda_F} (\lambda+m)^{N_f} e^{-\bar{\beta}\lambda} \quad (56)$$

An analogous definition is obviously possible for arbitrary coherent LQCD since the total action can always be eigenmode-expanded in this case.

We finally wish to make the following two comments.

(i) It should be noted that it is not at all a priori obvious to what extent are the above two ways of defining effective LQCD connected to one another. However, it is quite pleasing that the basic concept we are aiming at with effective LQCD can possibly be defined at the action level.

(ii) While not entirely straightforward, it is not unreasonable to believe that the effective theory could be directly simulated. It would be quite intriguing indeed to find out what kind of typical configurations would appear and how they compare to configurations obtained in effective LQCD I.

## 8 Conclusions

In this work we have presented a novel point of view at the process of constructing lattice regularizations. Considerations of universality make it possible to choose lattice dynamics from a very large set of possibilities (actions). This fact offers the advantage in that we can select the lattice theory that suits a particular problem and/or computational resources available. For example, large-scale lattice simulations are in majority of cases performed with simplest actions since it is judged that, given that they are amenable to fast simulation and define the correct theory in the continuum limit, it is efficient to use them even though

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<sup>12</sup>We emphasize that, strictly speaking, the continuum limit at fixed  $\Lambda_F$  does not necessarily have to exist. However, what is important for the concept of effective LQCD to be viable is that the continuum limit at the associated fixed momentum scale exists. These issues will be discussed in the third paper of this series.

they might have large cutoff effects. Taking the advantage of universality in this way, one typically treats the gauge and fermionic parts of the action independently of one another.

The main message of this article comes down to the suggestion that paying attention to the coherence between gauge and fermionic parts of lattice dynamics might be beneficial in certain circumstances. More precisely, we propose that the chirally symmetric Dirac operator  $D$  be the unifying element in the construction of lattice actions. This conclusion is based on two different motivations. (1) If one accepts that the physical content of gauge configuration  $U$  is closely tied (or identical) to the set of effective phases acquired by chiral fermion when hopping from  $n+\mu$  to  $n$  (principle of chiral ordering [1]), then the corresponding transformation  $U \rightarrow \mathcal{M}^D(U)$  drives lattice theory to the form where it is described by  $D$ . It is expected that configurations dominating the path integral in such theory will exhibit an increased degree of space-time order. (2) The formal definition for action in the continuum can be viewed as a matrix expression built entirely from  $D$  (see Eq. (44)), suggesting that, for construction of lattice actions,  $D$  can be viewed as a fundamental object.

We have proposed several formulations of *coherent* LQCD that respect this implied coherence, with the expectation that their use will make the transition to the continuum limit smoother at least for questions related to QCD vacuum structure. The simplest of these (such as theory described in Sec. 2, where  $S^G \propto (\text{Tr } D + \text{const})$ ) do not appear to pose major qualitative problems in terms of their inclusion into existing ways of simulating overlap fermions [7]. While this remains to be seen, it is quite clear already that there is a conceptual value in these formulations since they can make explicit certain aspects of QCD dynamics that are otherwise masked in incoherent formulations. The natural direction in this regard is to explore possible connections between gauge and fermionic aspects of the theory. This route was followed to some extent in this paper via proposing a *logarithmic* LQCD (Sec. 3), and especially *symmetric logarithmic* LQCD (Sec. 4). Here quarks and gluons contribute to the overall dynamics in a completely form-symmetric manner as expressed most clearly by Eq. (38). Indeed, the gluonic contribution can be viewed exactly as that of a quark that becomes infinitely heavy in the continuum limit. The issues related to locality properties of this theory were discussed in detail, and the way of resolving them was proposed.

We have argued that, in addition to the action defining the lattice theory, one can also build coherently (based on  $D$ ) all the operators of interest in QCD. This can be done in a generic way via the use of chiral ordering transformations. Moreover, the diagonal parts of various Clifford components associated with  $D$  offer explicit expressions for some of the useful composite fields (most notably  $F_{\mu\nu}$  in addition to scalar and pseudoscalar densities). Finally, a novel approach to definition of *effective* LQCD [1] at fermionic response scale  $\Lambda_F$  was proposed. Our approach relies on the fact that, in a coherent formulation, entire theory can typically be eigenmode-expanded thus providing the means for an explicit definition.

**Acknowledgments:** The author is grateful to Keh-Fei Liu for persistently reminding him over the course of past three years that this ancient idea should be published. Also, many thanks to Keh-Fei and Andrei Alexandru for numerous helpful discussions on this topic, and for the collaboration that was important for turning it into a realistic project. The author also benefited from discussions with Terry Draper, Al Shapere and Thomas Streuer, and from correspondence with Mike Creutz and Tony Kennedy. This work was supported by the U.S. Department of Energy under the grant DE-FG05-84ER40154.

## A Conventions in the Continuum

Here we summarize the conventions that fix the formal equations of QCD in the continuum. Gauge field  $A_\mu(x) \in \text{su}(3)$  is the vector field of traceless anti-Hermitian matrices. The associated field-strength tensor is given by

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] \quad (57)$$

while the covariant derivative acts via

$$D_\mu \phi(x) = (\partial_\mu + A_\mu(x)) \phi(x) \quad [D_\mu, D_\nu] \phi(x) = F_{\mu\nu}(x) \phi(x) \quad (58)$$

The gauge part of the full action is defined by

$$S^G = \int d^4x \left[ -\frac{1}{2g^2} \text{tr} F_{\mu\nu}(x) F_{\mu\nu}(x) + \frac{i\theta}{16\pi^2} \text{tr} F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x) \right] \quad (59)$$

where  $\tilde{F}_{\mu\nu}(x) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}(x)$ , while the fermionic part for single flavor of quarks reads

$$S^F = \int d^4x \bar{\psi}(x) (D_\mu \times \gamma_\mu + m) \psi(x) \quad (60)$$

The above actions are invariant under local gauge transformations that take the form

$$\begin{aligned} A_\mu(x) &\longrightarrow G(x) A_\mu(x) G^{-1}(x) + G(x) (\partial_\mu G^{-1}(x)) \\ \psi(x) &\longrightarrow (G(x) \times \mathbb{I}^s) \psi(x) \end{aligned} \quad (61)$$

where  $G(x) \in \text{SU}(3)$  specifies the transformation.

## B Fermionic Determinants

In this appendix, we emphasize a straightforward but noteworthy consequence of Conjecture C4. By construction, the gauge action of a given configuration in logarithmic LQCD is directly related to the fermionic determinant at a particular mass. Let us now apply the reverse logic and express the fermionic determinant for a smooth configuration in terms of its  $F^2$ . Thus, consider a formal expression for the fermionic determinant in the continuum

$$\det \left( D^{cont}(A) + m \right) \quad (62)$$

where  $A \equiv \{A_\mu(x)\}$  are fixed smooth gauge potentials on a finite torus of size  $L_p$ , and  $m$  is a fixed number. If we give meaning to the determinant by regularizing it on the hypercubic lattice using Dirac operator  $D \in \mathcal{S}^F$  then we have

$$\det \left( D(A, L) + m \right) = e^{\text{Tr} [\ln(D(A, L) + m) - \ln(D(0, L) + m)]} \det \left( D(0, L) + m \right) \quad (63)$$

where  $D(A, L) \equiv D(U(a_L))$  represents a Dirac operator on the lattice with  $L$  sites in each direction, and  $U(a_L)$  is a discretization of  $A$  according to prescription (2) with  $a_L \equiv L_p/L$ . Note that  $D(0, L)$  denotes the free lattice Dirac operator on this lattice. If we apply the Conjecture C4 in the above relation we obtain

$$\begin{aligned} \det \left( D^{cont}(A) + m \right) &\equiv \lim_{L \rightarrow \infty} \det \left( D(A, L) + m \right) \\ &= \lim_{L \rightarrow \infty} e^{-c^S(m) \sum_n a_L^4 \text{tr} F_{\mu\nu}(a_L n) F_{\mu\nu}(a_L n) + \mathcal{O}(a_L^6)} \det \left( D(0, L) + m \right) \\ &= e^{-c^S(m) \int d^4x \text{tr} F_{\mu\nu}(x) F_{\mu\nu}(x)} \det \left( D^{cont}(0) + m \right) \end{aligned} \quad (64)$$

Note that the above relation is still formal because the determinant of the free operator is not necessarily finite. However, what is physically relevant is the ratio of the determinants for two configurations  $A^{(1)}$  and  $A^{(2)}$ , where the free-field factor drops out at arbitrary  $L$ . We thus have

$$\frac{\det \left( D^{cont}(A^{(1)}) + m \right)}{\det \left( D^{cont}(A^{(2)}) + m \right)} = e^{-c^S(m) \int d^4x \text{tr} [F_{\mu\nu}^{(1)}(x) F_{\mu\nu}^{(1)}(x) - F_{\mu\nu}^{(2)}(x) F_{\mu\nu}^{(2)}(x)]} \quad (65)$$

where  $F_{\mu\nu}^{(1)}$  and  $F_{\mu\nu}^{(2)}$  are the field-strengths corresponding to  $A^{(1)}$  and  $A^{(2)}$ .

Now, instead of smooth gauge fields, consider arbitrary classical fields (i.e. with singularities allowed on the subset of space-time with measure zero), such that the corresponding  $F^2$  is Riemann-integrable over the torus in question. Then the required classical limits are still expected to exist (see point (i) in Sec. 2) and equation (65) will still be valid. Thus, regardless of mass  $m$ , the relative weight of two classical configurations in QCD path integral is completely determined by their  $F^2$  content if Conjecture C4 is valid.

## C Few Relevant Statements

For completeness (and because they are interesting in their own right) we state here a conjecture containing three ingredients that were implicitly used in the derivation of result (35). These statements relate to the existence and robust behavior of locality parameters  $A^T(a)$ ,  $\alpha^T(a)$  associated with the operator  $s_n(U, m_0) \equiv \text{tr} \ln(D(U) + m_0)$ . Thus, for the conjecture below we assume that  $s_n(U, m_0)$  is constructed using arbitrary  $D \in \mathcal{S}^F$  such that  $\ln(D + m_0)$  is well-defined for any  $m_0 > 0$ . Furthermore, we will consider the lattice action  $S(p)$  of symmetric logarithmic LQCD (22) for  $\theta = 0$ , in which case the set of bare lattice parameters consists of positive masses  $p \equiv \{m_0, m_1, m_2, \dots, m_{N_f}\}$  for arbitrary  $N_f \geq 0$ . Let us consider continuous paths  $p(t)$  ( $t \geq 0$ ) in this parameter space that approach  $p \equiv \{0, 0, \dots, 0\}$  monotonically in each component as  $t \rightarrow 0$ , i.e.  $m_j(0) = 0$  and  $m_j(t_1) < m_j(t_2)$  if  $t_1 < t_2$ , for all masses  $m_j$ . We will refer to such paths as *monotonic paths* for short. For arbitrary monotonic path we define the function  $\Delta_{n,m}^s(t)$  corresponding to operator  $s_n$  in the same way as in Eqs. (27,28), namely

$$\Delta_{n,m}^s(t) \equiv \left\langle \Delta_{n,m}^s(U, m_0(t)) \right\rangle_{S(p(t))} \quad (66)$$

We propose the following conjecture to be true

**Conjecture C8.** *Let  $p(t)$  be a monotonic path in parameter space of symmetric logarithmic LQCD with arbitrary number of flavors. Then the following statements hold.*

(i) *For arbitrary  $t > 0$  there exist positive numbers  $A(t)$ ,  $\alpha(t)$  such that*

$$\Delta_{n,m}^s(t) \leq A(t) e^{-\alpha(t)|n-m|} \quad \forall n, m \quad (67)$$

(ii) *If  $A(t)$  is optimal then the limit below exists and satisfies*

$$0 \leq \lim_{t \rightarrow 0} \frac{A(t)}{|c^S(m_0(t))|} < \infty \quad (68)$$

(iii) *If  $\alpha(t)$  is optimal then the limit below exists and satisfies*

$$0 < \lim_{t \rightarrow 0} \frac{\alpha(t)}{m_0(t)} < \infty \quad (69)$$

As before, by optimal pair  $A$ ,  $\alpha$  we mean maximal possible  $\alpha$  and then minimal  $A$  given that choice. Note that the part (i) of the above is a consequence of Conjecture C5 (strong locality of  $s_n$  at arbitrary fixed  $m_0$ ). Part (ii) asserts that the leading divergence in ensemble average of  $\text{tr} \ln(D + m_0)_{n,n}$  for  $m_0 \rightarrow 0$  can be removed via division by diverging  $c^S(m_0)$ . This conclusion is in fact stronger than what is needed for derivation of (35). The meaning of part (iii) is that the inverse range of locality for  $s_n$  in the vicinity of  $m_0 = 0$  is crucially driven by  $m_0$  not only for smooth configurations but also in the ensemble averages of symmetric logarithmic LQCD.



## References

- [1] I. Horváth, [hep-lat/0605008](#).
- [2] P. Hasenfratz, V. Laliena, F. Niedermayer, Phys. Lett. **B427** (1998) 125.
- [3] R. Narayanan and H. Neuberger, Nucl. Phys. **B443**, 305 (1995).
- [4] H. Neuberger, Phys. Lett. **B417** (1998) 141; Phys. Lett. **B427** (1998) 353.
- [5] Y. Kikukawa and A. Yamada, Phys. Lett. **B448** (1999) 265; K. Fujikawa, Nucl. Phys. **B546** (1999) 480; D. Adams, Annals Phys. **296** (2002) 131; H. Suzuki, Prog. Theor. Phys. **102** (1999) 141; D. Adams, J. Math. Phys. **42** (2001) 5522; W. Kerler, Int. J. Mod. Phys. **A16** (2001) 3117.
- [6] A. Alexandru, I. Horváth, K.F. Liu, in preparation.
- [7] K.F. Liu, A. Alexandru, I. Horváth, in preparation.
- [8] I. Horváth, Phys. Rev. Lett. **81** (1998) 4063; I. Horváth, Phys. Rev. **D60** (1999) 034510; I. Horváth, C.T. Balwe, R. Mendris, Nucl. Phys. **B599**, 283 (2001).
- [9] P. Hernández, K. Jansen, M. Lüscher, Nucl. Phys. **B552**, 363 (1999).
- [10] T. Draper *et al.*, Proc. Sci. **LAT2005** (2005) 120. [[hep-lat/0510075](#)]
- [11] I. Horváth, S.J. Dong, T. Draper, F.X. Lee, K.F. Liu, N. Mathur, H.B. Thacker, J.B. Zhang, Phys. Rev. **D68**, 114505 (2003).
- [12] A. Alexandru, I. Horváth, J.B. Zhang, Phys. Rev. **D72** (2005) 034506.
- [13] F. Niedermayer, Nucl. Phys. **B** (Proc. Suppl.) 73, 105 (1999).
- [14] K.F. Liu, A. Alexandru, I. Horváth, in preparation.
- [15] I. Horváth, S.J. Dong, T. Draper, F.X. Lee, K.F. Liu, H.B. Thacker, J.B. Zhang, Phys. Rev. **D67**, 011501(R) (2003).
- [16] I. Horváth, A. Alexandru, J.B. Zhang, S.J. Dong, T. Draper, F.X. Lee, K.F. Liu, N. Mathur, H.B. Thacker, Phys. Lett. **B612** (2005) 21.
- [17] F. Bruckmann, E.-M. Ilgenfritz, Phys. Rev. **D72**, 114502 (2005).
- [18] C. Gattringer, Phys. Rev. Lett. **88** 221601 (2002).